

Solutions of multi-component NLS models and spinor Bose-Einstein condensates

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Abstract

A three- and five-component nonlinear Schrodinger-type models, which describe spinor Bose-Einstein condensates (BEC's) with hyperfine structures $F = 1$ and $F = 2$ respectively, are studied. These models for particular values of the coupling constants are integrable by the inverse scattering method. They are related to symmetric spaces of **BD.I**-type $\simeq \text{SO}(2r+1)/\text{SO}(2) \times \text{SO}(2r-1)$ for $r = 2$ and $r = 3$. Using conveniently modified Zakharov-Shabat dressing procedure we obtain different types of soliton solutions.

Key words: Bose-Einstein condensates, integrable systems, soliton models

1 Introduction

The dynamics of spinor BECs is described by a three-component Gross-Pitaevskii (GP) system of equations. In the one-dimensional approximation the GP system goes into the following multicomponent nonlinear Schrödinger (MNLS) equation in 1D x -space [1]:

$$\begin{aligned} i\partial_t \Phi_1 + \partial_x^2 \Phi_1 + 2(|\Phi_1|^2 + 2|\Phi_0|^2)\Phi_1 + 2\Phi_{-1}^* \Phi_0^2 &= 0, \\ i\partial_t \Phi_0 + \partial_x^2 \Phi_0 + 2(|\Phi_{-1}|^2 + |\Phi_0|^2 + |\Phi_1|^2)\Phi_0 + 2\Phi_0^* \Phi_1 \Phi_{-1} &= 0, \\ i\partial_t \Phi_{-1} + \partial_x^2 \Phi_{-1} + 2(|\Phi_{-1}|^2 + 2|\Phi_0|^2)\Phi_{-1} + 2\Phi_1^* \Phi_0^2 &= 0. \end{aligned} \quad (1)$$

We consider BECs of alkali atoms in the $F = 1$ hyperfine state, elongated in x direction and confined in the transverse directions y, z by purely optical means. Thus the assembly of atoms in the $F = 1$ hyperfine state can be described by a normalized spinor wave vector $\Phi(x, t) = (\Phi_1(x, t), \Phi_0(x, t), \Phi_{-1}(x, t))^T$ whose components are labeled by the values of $m_F = 1, 0, -1$. The above model is integrable by means of the inverse scattering transform method [1].

It also allows an exact description of the dynamics and interaction of bright solitons with spin degrees of freedom. Matter-wave solitons are expected to be useful in atom laser, atom interferometry and coherent atom transport. It could contribute to the realization of quantum information processing or computation, as a part of new field of atom optics.

Lax pairs and geometrical interpretation of the MNLS models related to symmetric spaces (including the model (1)) are given in [2]. Darboux transformation for this special integrable model is developed in [3]. In [4] the authors study soliton solutions for the multicomponent Gross-Pitaevskii equation for $F = 2$ spinor condensate by two different methods assuming single-mode amplitudes and by generalizing Hirota's direct method for multicomponent systems. They point out the importance of integrable cases, which take place for particular choices of the coupling constants.

The aim of present paper is to show that both systems mentioned above are integrable by the inverse scattering method and are related to symmetric spaces [5] **BD.I**-type: $\simeq \text{SO}(2r+1)/\text{SO}(2) \times \text{SO}(2r-1)$ with $r = 2$ and $r = 3$ respectively. In Section 2 we formulate the Lax representations for the models. Section 3 is devoted to the $F = 2$ BEC model. In Section 4 we construct the fundamental analytic solutions of the corresponding Lax operator L and reduce the inverse scattering problem (ISP) for L to a Riemann-Hilbert problem (RHP). Using the special properties of the **BD.I** symmetric spaces we also obtain the minimal sets of scattering data \mathfrak{T}_i each of which allow one to reconstruct both the scattering matrix $T(\lambda)$ and the corresponding potential $Q(x, t)$. This allows us to derive in Section 5 their soliton solutions using suitable modification of the Zakharov-Shabat dressing method, proposed in [13,6].

2 Multicomponent nonlinear Schrödinger equations for **BD.I** series of symmetric spaces

MNLS equations for the **BD.I** series of symmetric spaces (algebras of the type $\mathfrak{so}(2r+1)$ and J dual to e_1) have the Lax representation $[L, M] = 0$ as follows

$$L\psi(x, t, \lambda) \equiv i\partial_x\psi + (Q(x, t) - \lambda J)\psi(x, t, \lambda) = 0. \quad (2)$$

$$M\psi(x, t, \lambda) \equiv i\partial_t\psi + (V_0(x, t) + \lambda V_1(x, t) - \lambda^2 J)\psi(x, t, \lambda) = 0, \quad (3)$$

$$V_1(x, t) = Q(x, t), \quad V_0(x, t) = i\text{ad}_J^{-1} \frac{dQ}{dx} + \frac{1}{2} [\text{ad}_J^{-1} Q, Q(x, t)]. \quad (4)$$

where

$$Q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0 \vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix}, \quad J = \text{diag}(1, 0, \dots, 0, -1). \quad (5)$$

The $2r - 1$ -vectors \vec{q} and \vec{p} have the form

$$\vec{q} = (q_2, \dots, q_r, q_{r+1}, q_{r+2}, \dots, q_{2r})^T, \quad \vec{p} = (p_2, \dots, p_r, p_{r+1}, p_{r+2}, \dots, p_{2r})^T,$$

while the matrix s_0 represents the metric involved in the definition of $so(2r-1)$, therefore it is related to the metric S_0 associated with $so(2r+1)$ in the following manner

$$S_0 = \sum_{k=1}^{2r+1} (-1)^{k+1} E_{k, 2r+2-k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (E_{kn})_{ij} = \delta_{ik} \delta_{nj} \quad (6)$$

Next we will use

$$\vec{E}_1^\pm = (E_{\pm(e_1-e_2)}, \dots, E_{\pm(e_1-e_r)}, E_{\pm e_1}, E_{\pm(e_1+e_r)}, \dots, E_{\pm(e_1+e_2)}), \quad (7)$$

We will use also the "scalar product"

$$(\vec{q} \cdot \vec{E}_1^+) = \sum_{k=2}^r (q_k(x, t) E_{e_1-e_k} + q_{2r-k+2}(x, t) E_{e_1+e_k}) + q_{r+1}(x, t) E_{e_1}.$$

Then the generic form of the potentials $Q(x, t)$ related to these type of symmetric spaces is

$$Q(x, t) = (\vec{q}(x, t) \cdot \vec{E}_1^+) + (\vec{p}(x, t) \cdot \vec{E}_1^-), \quad (8)$$

where E_α are the Weyl generators of the corresponding Lie algebra (see [5] for details) and Δ_1^+ is the set of all positive roots of $so(2r+1)$ such that $(\alpha, e_1) = 1$. In fact $\Delta_1^+ = \{e_1, e_1 \pm e_k, k = 2, \dots, r\}$.

In terms of these notations the generic MNLS type equations connected to **BD.I.** acquire the form

$$\begin{aligned} i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}, \vec{p})\vec{q} - (\vec{q}, s_0 \vec{q})s_0 \vec{p} &= 0, \\ i\vec{p}_t - \vec{p}_{xx} - 2(\vec{q}, \vec{p})\vec{p} - (\vec{p}, s_0 \vec{p})s_0 \vec{q} &= 0, \end{aligned} \quad (9)$$

In the case of $r = 2$ if we impose the reduction $p_k = q_k^*$ and introduce the new variables $\Phi_1 = q_2$, $\Phi_0 = q_3/\sqrt{2}$, $\Phi_{-1} = q_4$ then we reproduce the equations (1).

3 F=2 spinor Bose-Einstein condensate, integrable case

Let us introduce Hamiltonian for MNLS equations (9) with $\vec{p} = \epsilon \vec{q}^*$, $\epsilon = \pm 1$

$$H_{\text{MNLS}} = \int_{-\infty}^{\infty} dx \left((\partial_x \vec{q}, \partial_x \vec{q}^*) - \epsilon (\vec{q}, \vec{q}^*)^2 + \epsilon (\vec{q}, s_0 \vec{q}) (\vec{q}^*, s_0 \vec{q}^*) \right), \quad (10)$$

Define the number density and the singlet-pair amplitude by [7,8,4]

$$n = (\vec{\Phi}, \vec{\Phi}^*) = \sum_{\alpha=-2\dots 2} \Phi_{\alpha} \Phi_{\alpha}^*, \quad \Theta = (\vec{\Phi}, s_0 \vec{\Phi}). \quad (11)$$

where $\Phi_2 = q_2$, $\Phi_1 = q_3$, $\Phi_0 = q_4$, $\Phi_{-1} = q_5$, $\Phi_{-2} = q_6$. Then the singlet-pair amplitude take the form [7,8,4]

$$\Theta = 2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2. \quad (12)$$

The physical meaning of Θ is a measure of formation of spin-singlet "pairs" of bosons. The assembly of atoms in the $F = 2$ hyperfine state can be described by a normalized spinor wave vector

$$\Phi(x, t) = (\Phi_2(x, t), \Phi_1(x, t), \Phi_0(x, t), \Phi_{-1}(x, t), \Phi_{-2}(x, t))^T, \quad (13)$$

whose components are labeled by the values of $m_F = 2, 1, 0, -1, -2$. Here the energy functional within mean-field theory [9,10,7,8,4] is defined by

$$E_{\text{GP}}[\Phi] = \int_{-\infty}^{\infty} dx \left(\frac{\hbar^2}{2m} |\partial_x \Phi|^2 + \frac{c_0}{2} n^2 + \frac{c_2}{2} \mathbf{f}^2 + \frac{c_4}{2} |\Theta|^2 \right). \quad (14)$$

The coupling constants c_i are real and can be expressed in terms of a transverse confinement radius and a linear combination of the s -wave scattering lengths of atoms [1,11,12] and \mathbf{f} describe spin densities [4]. Choosing $c_2 = 0$, $c_4 = 1$ and $c_0 = -2$ we obtain integrable by the inverse scattering method model with the Hamiltonian. We set for simplicity $\hbar = 1$, $2m = 1$ without any loss of generality. The evolution equation is described by the multi-component Gross-Pitaevskii equation in one dimension [4]

$$i \frac{\partial \Phi}{\partial t} = \frac{\delta E_{\text{GP}}[\Phi]}{\delta \Phi^*}. \quad (15)$$

Then we have

$$i\vec{\Phi}_t + \vec{\Phi}_{xx} = -2\epsilon(\vec{\Phi}, \vec{\Phi}^*)\vec{\Phi} + \epsilon(\vec{\Phi}, s_0\vec{\Phi})s_0\vec{\Phi}^*, \quad (16)$$

or in explicit form by components we have

$$\begin{aligned} i\partial_t\Phi_{\pm 2} + \partial_{xx}\Phi_{\pm 2} &= -2\epsilon(\vec{\Phi}, \vec{\Phi}^*)\Phi_{\pm 2} + \epsilon(2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2)\Phi_{\mp 2}^*, \\ i\partial_t\Phi_{\pm 1} + \partial_{xx}\Phi_{\pm 1} &= -2\epsilon(\vec{\Phi}, \vec{\Phi}^*)\Phi_{\pm 1} - \epsilon(2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2)\Phi_{\mp 1}^*, \\ i\partial_t\Phi_0 + \partial_{xx}\Phi_0 &= -2\epsilon(\vec{\Phi}, \vec{\Phi}^*)\Phi_0 + \epsilon(2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2)\Phi_0^*. \end{aligned}$$

4 Inverse scattering method and reconstruction of potential from minimal scattering data

Herein we remind some basic features of the inverse scattering theory appropriate for the special case of $F = 2$ spinor BEC equations.

Solving the direct and the inverse scattering problem (ISP) for L uses the Jost solutions which are defined by, see [16] and the references therein

$$\lim_{x \rightarrow -\infty} \phi(x, t, \lambda) e^{i\lambda Jx} = \mathbb{1}, \quad \lim_{x \rightarrow \infty} \psi(x, t, \lambda) e^{i\lambda Jx} = \mathbb{1} \quad (17)$$

and the scattering matrix $T(\lambda, t) \equiv \psi^{-1}\phi(x, t, \lambda)$. Due to the special choice of J and to the fact that the Jost solutions and the scattering matrix take values in the group $SO(2r+1)$ we can use the following block-matrix structure of $T(\lambda, t)$

$$T(\lambda, t) = \begin{pmatrix} m_1^+ & -\vec{b}^{-T} & c_1^- \\ \vec{b}^+ & \mathbf{T}_{22} & -s_0\vec{b}^- \\ c_1^+ & \vec{b}^{+T}s_0 & m_1^- \end{pmatrix}, \quad (18)$$

where $\vec{b}^\pm(\lambda, t)$ are $2r-1$ -component vectors, $\mathbf{T}_{22}(\lambda)$ is a $2r-1 \times 2r-1$ block and $m_1^\pm(\lambda)$, $c_1^\pm(\lambda)$ are scalar functions satisfying $c_1^\pm = 1/2(\vec{b}^\pm \cdot s_0\vec{b}^\pm)/m_1^\pm$.

Important tools for reducing the ISP to a Riemann-Hilbert problem (RHP) are the fundamental analytic solution (FAS) $\chi^\pm(x, t, \lambda)$. Their construction is based on the generalized Gauss decomposition of $T(\lambda, t)$

$$\chi^\pm(x, t, \lambda) = \phi(x, t, \lambda)S_J^\pm(t, \lambda) = \psi(x, t, \lambda)T_J^\mp(t, \lambda)D_J^\pm(\lambda). \quad (19)$$

Here S_J^\pm , T_J^\pm upper- and lower- block-triangular matrices, while $D_J^\pm(\lambda)$ are block-diagonal matrices with the same block structure as $T(\lambda, t)$ above. Skipping the details we give the explicit expressions of the Gauss factors in terms of the matrix elements of $T(\lambda, t)$

$$S_J^\pm(t, \lambda) = \exp\left(\pm(\vec{\tau}^\pm(\lambda, t) \cdot \vec{E}_1^\pm)\right), \quad T_J^\pm(t, \lambda) = \exp\left(\mp(\vec{\rho}^\pm(\lambda, t) \cdot \vec{E}_1^\pm)\right),$$

$$D_J^+ = \begin{pmatrix} m_1^+ & 0 & 0 \\ 0 & \mathbf{m}_2^+ & 0 \\ 0 & 0 & 1/m_1^+ \end{pmatrix}, \quad D_J^- = \begin{pmatrix} 1/m_1^- & 0 & 0 \\ 0 & \mathbf{m}_2^- & 0 \\ 0 & 0 & m_1^- \end{pmatrix}, \quad (20)$$

where $\vec{\tau}^\pm(\lambda, t) = \vec{b}^\mp/m_1^\pm$, $\vec{\rho}^\pm(\lambda, t) = \vec{b}^\pm/m_1^\pm$ and

$$\mathbf{m}_2^+ = \mathbf{T}_{22} + \frac{\vec{b}^+ \vec{b}^{-T}}{m_1^+}, \quad \mathbf{m}_2^- = \mathbf{T}_{22} + \frac{s_0 \vec{b}^- \vec{b}^{+T} s_0}{m_1^-}.$$

If $Q(x, t)$ evolves according to (1) then the scattering matrix and its elements satisfy the following linear evolution equations

$$i \frac{d\vec{b}^\pm}{dt} \pm \lambda^2 \vec{b}^\pm(t, \lambda) = 0, \quad i \frac{dm_1^\pm}{dt} = 0, \quad i \frac{d\mathbf{m}_2^\pm}{dt} = 0, \quad (21)$$

so the block-diagonal matrices $D^\pm(\lambda)$ can be considered as generating functionals of the integrals of motion. The fact that all $(2r-1)^2$ matrix elements of $m_2^\pm(\lambda)$ for $\lambda \in \mathbb{C}_\pm$ generate integrals of motion reflect the superintegrability of the model and are due to the degeneracy of the dispersion law of (1). We remind that $D_J^\pm(\lambda)$ allow analytic extension for $\lambda \in \mathbb{C}_\pm$ and that their zeroes and poles determine the discrete eigenvalues of L .

The FAS for real λ are linearly related

$$\chi^+(x, t, \lambda) = \chi^-(x, t, \lambda) G_J(\lambda, t), \quad G_{0,J}(\lambda, t) = S_J^-(\lambda, t) S_J^+(\lambda, t). \quad (22)$$

One can rewrite eq. (22) in an equivalent form for the FAS $\xi^\pm(x, t, \lambda) = \chi^\pm(x, t, \lambda) e^{i\lambda Jx}$ which satisfy also the relation

$$\lim_{\lambda \rightarrow \infty} \xi^\pm(x, t, \lambda) = \mathbb{1}. \quad (23)$$

Then these FAS satisfy

$$\xi^+(x, t, \lambda) = \xi^-(x, t, \lambda) G_J(x, \lambda, t), \quad G_J(x, \lambda, t) = e^{-i\lambda Jx} G_{0,J}^-(\lambda, t) e^{i\lambda Jx}. \quad (24)$$

Obviously the sewing function $G_J(x, \lambda, t)$ is uniquely determined by the Gauss factors $S_J^\pm(\lambda, t)$. In view of eq. (20) we arrive to the following

Lemma 1. *Let the potential $Q(x, t)$ is such that the Lax operator L has no discrete eigenvalues. Then as minimal set of scattering data which determines uniquely the scattering matrix $T(\lambda, t)$ and the corresponding potential $Q(x, t)$ one can consider either one of the sets \mathfrak{T}_i , $i = 1, 2$*

$$\mathfrak{T}_1 \equiv \{\vec{\rho}^+(\lambda, t), \vec{\rho}^-(\lambda, t), \quad \lambda \in \mathbb{R}\}, \quad \mathfrak{T}_2 \equiv \{\vec{\tau}^+(\lambda, t), \vec{\tau}^-(\lambda, t), \quad \lambda \in \mathbb{R}\}. \quad (25)$$

Proof. i) From the fact that $T(\lambda, t) \in SO(2r + 1)$ one can derive that

$$\frac{1}{m_1^+ m_1^-} = 1 + (\vec{\rho}^+, \vec{\rho}^-) + \frac{1}{4}(\vec{\rho}^+, s_0 \vec{\rho}^+)(\vec{\rho}^-, s_0 \vec{\rho}^-) \quad (26)$$

for $\lambda \in \mathbb{R}$. Using the analyticity properties of m_1^\pm we can recover them from eq. (26) using Cauchy-Plemelji formulae. Given \mathfrak{T}_i and m_1^\pm one easily recovers $\vec{b}^\pm(\lambda)$ and $c_1^\pm(\lambda)$. In order to recover \mathbf{m}_2^\pm one again uses their analyticity properties, only now the problem reduces to a RHP for functions on $SO(2r+1)$. The details will be presented elsewhere.

ii) Obviously, given \mathfrak{T}_i one uniquely recovers the sewing function $G_J(x, t, \lambda)$. In order to recover the corresponding potential $Q(x, t)$ one can use the fact that the RHP (24) with canonical normalization has unique solution. Given that solution $\chi^\pm(x, t, \lambda)$ one recovers $Q(x, t)$ via the formula

$$Q(x, t) = \lim_{\lambda \rightarrow \infty} \lambda \left(J - \chi^\pm J \hat{\chi}^\pm(x, t, \lambda) \right). \quad (27)$$

which is well known. \square

We impose also the standard reduction, namely assume that $Q(x, t) = Q^\dagger(x, t)$, or in components $p_k = q_k^*$. As a consequence we have $\vec{\rho}^-(\lambda, t) = \vec{\rho}^{+,*}(\lambda, t)$ and $\vec{\tau}^-(\lambda, t) = \vec{\tau}^{+,*}(\lambda, t)$.

5 Dressing method and soliton solutions

The main goal of the dressing method [17,18,19,20,21] is, starting from a known solutions $\chi_0^\pm(x, t, \lambda)$ of $L_0(\lambda)$ with potential $Q_{(0)}(x, t)$ to construct new singular solutions $\chi_1^\pm(x, t, \lambda)$ of L with a potential $Q_{(1)}(x, t)$ with two additinal singularities located at prescribed positions λ_1^\pm ; the reduction $\vec{p} = \vec{q}^*$ ensures that $\lambda_1^- = (\lambda_1^+)^*$. It is related to the regular one by a dressing factor $u(x, t, \lambda)$

$$\chi_1^\pm(x, t, \lambda) = u(x, \lambda) \chi_0^\pm(x, t, \lambda) u_-^{-1}(\lambda). \quad u_-(\lambda) = \lim_{x \rightarrow -\infty} u(x, \lambda) \quad (28)$$

Note that $u_-(\lambda)$ is a block-diagonal matrix. The dressing factor $u(x, \lambda)$ must satisfy the equation

$$i\partial_x u + Q_{(1)}(x)u - uQ_{(0)}(x) - \lambda[J, u(x, \lambda)] = 0, \quad (29)$$

and the normalization condition $\lim_{\lambda \rightarrow \infty} u(x, \lambda) = \mathbb{1}$. Besides $\chi_i^\pm(x, \lambda)$, $i = 0, 1$ and $u(x, \lambda)$ must belong to the corresponding Lie group $SO(2r + 1, \mathbb{C})$; in addition $u(x, \lambda)$ by construction has poles and zeroes at λ_1^\pm .

The construction of $u(x, \lambda)$ is based on an appropriate anzats specifying explicitly the form of its λ -dependence [6,20] and the references therein.

$$u(x, \lambda) = \mathbb{1} + (c(\lambda) - 1)P(x, t) + \left(\frac{1}{c(\lambda)} - 1\right)\overline{P}(x, t), \quad \overline{P} = S_0^{-1}P^T S_0, \quad (30)$$

where $P(x, t)$ and $\overline{P}(x, t)$ are projectors whose rank s can not exceed r and which satisfy $P\overline{P}(x, t) = 0$. Given a set of s linearly independent polarization vectors $|n_k\rangle$ spanning the corresponding eigensubspace of L one can define

$$P(x, t) = \sum_{a,b=1}^s |n_a(x, t)\rangle M_{ab}^{-1} \langle n_b^\dagger(x, t)|, \quad M_{ab}(x, t) = \langle n_b^\dagger(x, t)|n_a(x, t)\rangle, \quad (31)$$

$$|n_a(x, t)\rangle = \chi_0^+(x, t, \lambda^+) |n_{0,a}\rangle, \quad c(\lambda) = \frac{\lambda - \lambda^+}{\lambda - \lambda^-}, \quad \langle n_{0,a}|S_0|n_{0,b}\rangle = 0.$$

Taking the limit $\lambda \rightarrow \infty$ in eq. (29) we get that

$$Q_{(1)}(x, t) - Q_{(0)}(x, t) = (\lambda_1^- - \lambda_1^+) [J, P(x, t) - \overline{P}(x, t)].$$

Below we list the explicit expressions only for the one-soliton solutions. To this end we assume $Q_{(0)} = 0$ and put $\lambda_1^\pm = \mu \pm i\nu$. As a result we get

$$q_k^{(1s)}(x, t) = -2i\nu \left(P_{1k}(x, t) + (-1)^k P_{\bar{k}, 2r+1}(x, t) \right), \quad (32)$$

where $\bar{k} = 2r + 2 - k$.

Repeating the above procedure N times we can obtain N soliton solutions.

5.1 The case of rank one solitons

In this case $s = 1$ so that the generic (arbitrary r) one-soliton solution reads

$$q_k = \frac{-i\nu e^{-i\mu(x-vt-\delta_0)}}{\cosh 2z + \Delta_0^2} \left(\alpha_k e^{z-i\phi_k} + (-1)^k \alpha_{\bar{k}} e^{-z+i\phi_{\bar{k}}} \right),$$

$$v = \frac{\nu^2 - \mu^2}{\mu}, \quad u = -2\mu, \quad z(x, t) = \nu(x - ut - \xi_0), \quad (33)$$

$$\xi_0 = \frac{1}{2\nu} \ln \frac{|n_{0,2r+1}|}{|n_{0,1}|}, \quad \alpha_k = \frac{|n_{0,k}|}{\sqrt{|n_{0,1}||n_{0,2r+1}|}}, \quad \Delta_0^2 = \frac{\sum_{k=2}^{2r} |n_{0,k}|^2}{2|n_{0,1}n_{0,2r+1}|},$$

and $\delta_0 = \arg n_{0,1}/\mu = -\arg n_{0,2r+1}/\mu$, $\phi_k = \arg n_{0,k}$. The polarization vectors satisfy the following relation

$$\sum_{k=1}^r 2(-1)^{k+1} n_{0,k} n_{0,\bar{k}} + (-1)^r n_{0,r+1}^2 = 0. \quad (34)$$

Thus for $r = 2$ we identify $\Phi_1 = q_2$, $\Phi_0 = q_3/\sqrt{2}$ and $\Phi_3 = q_4$ and we obtain the following solutions for the equation (1)

$$\begin{aligned}\Phi_{\pm 1} &= -\frac{2i\nu\sqrt{\alpha_2\alpha_4}e^{-i\mu(x-vt-\delta_{\pm 1})}}{\cosh 2z + \Delta_0^2} (\cos \phi_{\pm 1} \cosh z_{\pm 1} - i \sin \phi_{\pm 1} \sinh z_{\pm 1}) \quad (35) \\ \delta_{\pm 1} &= \delta_0 \mp \frac{\phi_2 - \phi_4}{2\mu}, \quad \phi_{\pm 1} = \frac{\phi_2 + \phi_4}{2} \quad z_{\pm 1} = z \mp \frac{1}{2} \ln \frac{\alpha_4}{\alpha_2}, \\ \Phi_0 &= -\frac{\sqrt{2}i\nu\alpha_3 e^{-i\mu(x-vt-\delta_0)}}{\cosh 2z + \Delta_0^2} (\cos \phi_3 \sinh z - i \sin \phi_3 \cosh z). \quad (36)\end{aligned}$$

For $r = 3$ we identify $\Phi_2 = q_2$, $\Phi_1 = q_3$, $\Phi_0 = q_4$, $\Phi_{-1} = q_5$ and $\Phi_{-2} = q_6$, so that the one-soliton solution for equation (16) reads

$$\begin{aligned}\Phi_{\pm 2} &= -\frac{2i\nu\sqrt{\alpha_2\alpha_6}e^{-i\mu(x-vt-\delta_{\pm 2})}}{\cosh 2z + \Delta_0^2} (\cos \phi_{\pm 2} \cosh z_{\pm 2} - i \sin \phi_{\pm 2} \sinh z_{\pm 2}) \quad (37) \\ \Phi_{\pm 1} &= -\frac{2i\nu\sqrt{\alpha_3\alpha_5}e^{-i\mu(x-vt-\delta_{\pm 1})}}{\cosh 2z + \Delta_0^2} (\cos \phi_{\pm 1} \sinh z_{\pm 1} - i \sin \phi_{\pm 1} \cosh z_{\pm 1}) \quad (38) \\ \delta_{\pm 2} &= \delta_0 \mp \frac{\phi_2 - \phi_6}{2\mu}, \quad \phi_{\pm 2} = \frac{\phi_2 + \phi_6}{2} \quad z_{\pm 2} = z \mp \frac{1}{2} \ln \frac{\alpha_6}{\alpha_2}, \\ \delta_{\pm 1} &= \delta_0 \mp \frac{\phi_3 - \phi_5}{2\mu}, \quad \phi_{\pm 1} = \frac{\phi_3 + \phi_5}{2}, \quad z_{\pm 1} = z \mp \frac{1}{2} \ln \frac{\alpha_5}{\alpha_3}, \\ \Phi_0 &= -\frac{2i\nu\alpha_4 e^{-i\mu(x-vt-\delta_0)}}{\cosh 2z + \Delta_0^2} (\cos \phi_4 \cosh z - i \sin \phi_4 \sinh z). \quad (39)\end{aligned}$$

Choosing appropriately the polarization vectors $|n\rangle$ we are able to reproduce the soliton solutions obtained by Wadati et al. both for $F = 1$ and $F = 2$ BEC.

5.2 The case of rank two solitons

Here $s = 2$ and we have two linearly independent polarization vectors $|n_a\rangle$, $a = 1, 2$. From eq. (31) we get

$$\begin{aligned}P(x, t) &= \frac{1}{\det M} \left(|n_1(x, t)\rangle M_{22} \langle n_1^\dagger(x, t)| - |n_2(x, t)\rangle M_{12} \langle n_1^\dagger(x, t)| \right. \\ &\quad \left. - |n_1(x, t)\rangle M_{21} \langle n_2^\dagger(x, t)| + |n_2(x, t)\rangle M_{11} \langle n_2^\dagger(x, t)| \right), \quad (40) \\ \det M(x, t) &= M_{11}M_{22} - M_{12}M_{21}, \quad M_{ab}(x, t) = \langle n_a^\dagger(x, t)|n_b(x, t)\rangle,\end{aligned}$$

The corresponding expressions for the rank 2 soliton solution are obtained by inserting eq. (40) into (32) and are rather involved. We remark here that the

reduction $Q^\dagger = Q$ may not be sufficient to ensure that $\det M$ is positive for all x and t , so for certain choices of $|n_a\rangle$ we may have singular solitons. These and other properties of the rank 2 soliton solutions will be analyzed elsewhere.

6 Conclusions and discussion

The main result of the present paper is that a special version of the model describing $F = 2$ spinor Bose-Einstein condensate is integrable by the ISM. The corresponding Lax representation is naturally related to the symmetric space **BD.I.** $\simeq \text{SO}(7)/\text{SO}(2) \times \text{SO}(5)$, see [5]. For a generic hyperfine spin F , the dynamics within the mean field theory is described by the $2F + 1$ component Gross-Pitaevskii equation in one dimension. If all the spin dependent interactions vanish and only intensity interaction exists, the multi-component Gross-Pitaevskii equation in one dimension is equivalent to the vector nonlinear Schrödinger equation with $2F + 1$ components [22].

Then equations (9) with the reduction $p = \epsilon q^*, \epsilon = \pm 1$ are natural generalization of the vector nonlinear Schrödinger equation, which adequately model the spinor Bose-Einstein condensates for values of $F = r$ equal to 1 and 2. We expect that for generic F these equations may be useful in describing BECs with higher hyperfine structure.

Here we derived only generic one-soliton solutions. Following the ideas of [23] one can classify different types of one-soliton solutions related to different possible choices of the rank of $P(x, t)$ and its polarization vectors. One can also derive the N -soliton solutions by either repeating N times the dressing with u (see eq. (30)), or considering more general dressing factors u with $2N$ zeroes and poles in λ . These and other problems will be addressed elsewhere.

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